MacWilliams Extension Theorem for the Lee Weight

Noncommutative rings and their applications V
Lens 12-15 June 2017

Philippe Langevin

IMATH, Toulon

last revision June 11, 2017.

A serie of joint works with Serhii Dyshko and Jay Wood.

Sommaire

Isometry

- Let K be a finite field
- $\mathrm{H}(x) = \begin{cases} 0, & x = 0; \\ 1, & \text{else.} \end{cases}$
- n a positive integer
- C a subspace of K^n

Isometry

- Let K be a finite field
- n a positive integer
- $\mathrm{H}(x) = \begin{cases} 0, & x = 0; \\ 1, & \text{else.} \end{cases}$
- C a subspace of K^n

The Hamming weight of $x \in K^n$

$$\mathrm{w_H}(x) = \{i \mid x_i \neq 0\} = \sum_{i=1}^{n} \mathrm{H}(x_i),$$

Isometry

- Let K be a finite field
- n a positive integer
- $\mathrm{H}(x) = \begin{cases} 0, & x = 0; \\ 1, & \text{else.} \end{cases}$
- C a subspace of K^n

The Hamming weight of $x \in K^n$

$$\mathrm{w_H}(x) = \{i \mid x_i \neq 0\} = \sum_{i=1}^{n} \mathrm{H}(x_i),$$

A linear map $f\colon C \to K^n$ preserving the Hamming weight

$$\forall x \in C, \quad \mathrm{w_H}(x) = \mathrm{w_H}(f(x))$$

is called a (linear) *isometry* over C.

Monomial transformation

- consider $(e_i)_{1\leq i\leq n}$ the canonical basis of K^n.

An isometry over the ambiant space K^n permutes the vectors of weight one.

$$e_i \mapsto \lambda_i e_{\pi(i)}$$

where

- $\lambda_i \in K^\times$
- π permutes $\{1, 2, \ldots, n\}$.

Monomial transformation

- consider $(e_i)_{1\leq i\leq n}$ the canonical basis of K^n.

An isometry over the ambiant space K^n permutes the vectors of weight one.

$$e_i \mapsto \lambda_i e_{\pi(i)}$$

where

- $\lambda_i \in K^\times$
- π permutes $\{1, 2, \dots, n\}$.

$$x \mapsto (\lambda_1 x_{\pi(1)}, \lambda_2 x_{\pi(2)}, \dots, \lambda_n x_{\pi(n)})$$

often called a monomial transformation of K^n.

Monomial transformation

- consider $(e_i)_{1\leq i\leq n}$ the canonical basis of K^n.

An isometry over the ambiant space K^n permutes the vectors of weight one.

$$e_i \mapsto \lambda_i e_{\pi(i)}$$

where

- $\lambda_i \in K^\times$
- π permutes $\{1, 2, \ldots, n\}$.

$$x \mapsto (\lambda_1 x_{\pi(1)}, \lambda_2 x_{\pi(2)}, \ldots, \lambda_n x_{\pi(n)})$$

often called a monomial transformation of K^n.

$$\mathfrak{S}_n \ltimes {K^\times}^n$$

MacWilliams Extension Theorem

Theorem (MacWilliams, 1962)

An isometry over $C \subseteq K^n$ extends to an isometry over K^n.

MacWilliams Extension Theorem

Theorem (MacWilliams, 1962)

An isometry over $C \subseteq K^n$ extends to an isometry over K^n.

In other words, for an isometry $f \colon C \to K^n$ there exists a permutation π and scalars λ_i's such that

$$\forall x \in C, \quad f(x) = (\lambda_1 x_{\pi(1)}, \lambda_2 x_{\pi(2)}, \dots, \lambda_n x_{\pi(n)})$$

$$\mathfrak{S}_n \ltimes {K^{\times}}^n \xrightarrow{\text{res}} \mathrm{Isom(C)} \to 0$$

Frobenius ring case

From the character theorycal proof of Ward & Wood, one deduces that MacWilliams extension theorem works for the Hamming space over any finite Frobenius rings.

H. N. Ward, J. A. Wood, *Characters and the Equivalence of Codes*, J. Comb. Theory, Ser. A, (1996).

Homogeneous weight

The same holds for any homogeneous weight on a finite Frobenius ring :

- $\omega(0) = 0$;
- If x and y are associate then $\omega(x) = \omega(y)$;
- There exists a constant c such that for all principal ideal $\mathfrak{I}$,

$$\sum_{y \in I} \omega(y) = c \, |\mathfrak{I}| \, .$$

M. Greferath and S. E. Schmidt, *Finite-ring combinatorics and MacWilliams's equivalence theorem*, J. Combin. Theory Ser. A, (2000).

Homogeneous weight

The same holds for any homogeneous weight on a finite Frobenius ring :

- $\omega(0) = 0$;
- If x and y are associate then $\omega(x) = \omega(y)$;
- There exists a constant c such that for all principal ideal $\mathfrak{I}$,

$$\sum_{y \in I} \omega(y) = c\,|\mathfrak{I}|\,.$$

M. Greferath and S. E. Schmidt, *Finite-ring combinatorics and MacWilliams's equivalence theorem*, J. Combin. Theory Ser. A, (2000).

Of course, MacWilliams extension works over the $\mathbb{Z}/(4)$ with its Lee weight

$$\mathrm{L}(0) = 0, \quad \mathrm{L}(1) = \mathrm{L}(3) = 1, \quad \mathrm{L}(2) = 2.$$

MacWilliams for Lee weight

- q a positive integer
- L the Lee weight over $\mathbb{Z}/(q)$.

$$\mathrm{L}(r) = \begin{cases} r, & 0 \leq r \leq q/2; \\ q - r, & q/2 < r < q. \end{cases}$$

Remark

Lee weight is not homogeneous for $q > 4$.

Do we have a MacWilliams extension statement for the Lee weight ?

A. Barra, *Equivalence Theorems and the Local-Global Property*, ProQuest LLC, Ann Arbor, MI, 2012, Thesis (Ph.D.)–University of Kentucky.

Known results, new results

In the last NCRA IV proceedings :

- $q = 2p + 1$, p prime (Folklore).
- $q = 4p + 1$ (Barra, 2012)
- $q = 2^r$ or $q = 3^r$ (Lens, 2015)

Despite all this progress, there are glaring gaps in our knowledge : does extension theorem holds for linear codes over $\mathbb{Z}/(q)$?

Known results, new results

In the last NCRA IV proceedings :

- $q = 2p + 1$, p prime (Folklore).
- $q = 4p + 1$ (Barra, 2012)
- $q = 2^r$ or $q = 3^r$ (Lens, 2015)

Despite all this progress, there are glaring gaps in our knowledge : does extension theorem holds for linear codes over $\mathbb{Z}/(q)$?

YES !

Connection with classical tools

We have two ways to prove MacWilliams extension Theorem for the Lee weight using classical results of

1. Number Theory
2. Harmonic Analysis

Connection with classical tools

We have two ways to prove MacWilliams extension Theorem for the Lee weight using classical results of

1. Number Theory
2. Harmonic Analysis

The first works when the module q is primary, the second due to Sergey Dyshko works for a general module.

Connection with classical tools

We have two ways to prove MacWilliams extension Theorem for the Lee weight using classical results of

1. Number Theory
2. Harmonic Analysis

The first works when the module q is primary, the second due to Sergey Dyshko works for a general module.

I will sketch the proofs in the case of prime fields.

Extension property holds for Lee weight

- *Deux analogues au déterminant de Maillet* C. R. Acad. Sci. Paris vol. Ser. I, 2016
- Ph. Langevin, J. Wood: *The extension problem for Lee and Euclidean weights* Journal of Algebra Combinatorics Discrete Structures and Applications Vol. **4** 2 pp 207–217, 2017.
- Ph. Langevin, J. Wood: *The extension theorem for the Lee and Euclidean Weight over Z/p^kZ* Journal of Pure and Applied Algebra, submitted 2016.
- S. Dyshko: *The Extension Theorem for the Lee weight* Code, Design and Cryptography, submitted 2017 .

Sommaire

Isometry in general

- Let R be a finite ring
- ω a weight function on R
- n a positive integer
- M a submodule of R^n

Isometry in general

- Let R be a finite ring
- ω a weight function on R
- n a positive integer
- M a submodule of R^n

ω is a real function such that $\omega(r) = 0$ iff $r = 0$.

Isometry in general

- Let R be a finite ring
- ω a weight function on R
- n a positive integer
- M a submodule of R^n

ω is a real function such that $\omega(r) = 0$ iff $r = 0$.

The ω-weight of $x \in R^n$

$$\mathrm{w}_\omega(x) = \sum_{i=1}^{n} \omega(x_i)$$

Isometry in general

- Let R be a finite ring
- ω a weight function on R
- n a positive integer
- M a submodule of R^n

ω is a real function such that $\omega(r) = 0$ iff $r = 0$.

The ω-weight of $x \in R^n$

$$\mathrm{w}_\omega(x) = \sum_{i=1}^{n} \omega(x_i)$$

A linear map $f\colon M \to K^n$ preserving the ω-weight

$$\forall x \in M, \quad \omega(x) = \omega(f(x))$$

is called a (linear) ω-*isometry* over M.

U-monomial map

- e_i the canonical basis of R^n.

Again, an isometry over R^n maps e_i on $\lambda_i e_{\pi(i)}$ where $\lambda_i \in R^\times$ and π permutes $\{1, 2, \ldots, n\}$, moreover :

$$\forall t \in R, \quad \omega(t) = \mathrm{w}_\omega(te_i) = \mathrm{w}_\omega(t\lambda_i e_{\pi(i)}) = \omega(t\lambda_i)$$

thus λ_i lies in the symmetry group of ω

$$U(\omega) := \{\lambda \in R \mid \forall t \in R, \quad \omega(\lambda t) = \omega(t)\}$$

U-monomial map

- e_i the canonical basis of R^n.

Again, an isometry over R^n maps e_i on $\lambda_i e_{\pi(i)}$ where $\lambda_i \in R^\times$ and π permutes $\{1, 2, \ldots, n\}$, moreover :

$$\forall t \in R, \quad \omega(t) = \mathrm{w}_\omega(te_i) = \mathrm{w}_\omega(t\lambda_i e_{\pi(i)}) = \omega(t\lambda_i)$$

thus λ_i lies in the symmetry group of ω

$$U(\omega) := \{\lambda \in R \mid \forall t \in R, \quad \omega(\lambda t) = \omega(t)\}$$

Definition (U-monomial transformation)

Given U a subgroup of $R^\times$, a monomial transformation with scalars in U.

$$\mathfrak{S}_n \ltimes U^n$$

Extension Property

Definition (extension property)

We say that Extension Property holds for the pair (R, ω) when each ω-isometry over $M \subseteq R^n$ extends to a $U(\omega)$-monomial transformation.

Extension Property

Definition (extension property)

We say that Extension Property holds for the pair (R, ω) when each ω-isometry over $M \subseteq R^n$ extends to a $U(\omega)$-monomial transformation.

- EP holds for Hamming weight on Frobenius ring
- EP holds for Homogeneous weight on Frobenius ring

Extension Property

Definition (extension property)

We say that Extension Property holds for the pair (R, ω) when each ω-isometry over $M \subseteq R^n$ extends to a $U(\omega)$-monomial transformation.

- EP holds for Hamming weight on Frobenius ring
- EP holds for Homogeneous weight on Frobenius ring

It looks difficult to decide if EP holds for an arbitrary weight function!

Sommaire

Preserving map

- U be a subgroup of $R^\times$
- $r \sim s$ if and only if $s \in rU$
- Ω a set of representatives of $R \setminus \{0\}$
- $c_r(x) := \sharp\{i \mid x_i = r\}$
- $c_r^U(x) := \sharp\{i \mid x_i \sim r\}$

Preserving map

- U be a subgroup of $R^\times$
- $r \sim s$ if and only if $s \in rU$
- Ω a set of representatives of $R \setminus \{0\}$
- $c_r(x) := \sharp\{i \mid x_i = r\}$
- $c_r^U(x) := \sharp\{i \mid x_i \sim r\}$

A linear map $f\colon M \to R^n$ such that

$$\forall x \in C, \forall r \in \Omega \quad c_r^U(x) = c_r^U(f(x))$$

is called a *U-preserving* map over M.

Goldberg Extension Theorem

preserving map over K^n

The U-preserving maps over K^n are precisely the U-monomial transformations.

Goldberg Extension Theorem

preserving map over K^n

The U-preserving maps over K^n are precisely the U-monomial transformations.

Theorem (Goldberg, 1980)

A linear U-preserving map extends to U-monomial transformation.

Goldberg Extension Theorem

preserving map over K^n

The U-preserving maps over K^n are precisely the U-monomial transformations.

Theorem (Goldberg, 1980)

A linear U-preserving map extends to U-monomial transformation.

The same holds modular rings : Constantinescu, Heise, Honold (1996).

J. A. Wood.
Weight functions and the extension theorem for linear codes over finite rings.
In R. C. Mullin and G. L. Mullen, editors, *Finite fields: theory, applications, and algorithms (Waterloo, ON, 1997)*, volume 225 of *Contemp. Math.*, pages 231–243. Amer. Math. Soc., Providence, RI, 1999.

Extensibility Property (recall)

The symmetry group of ω.

$$U(\omega) = \{\lambda \in K^\times \mid \forall x \in K,\ \omega(\lambda x) = \omega(x)\} \leqslant K^\times$$

Extension Property

We say the extension property holds for the weight ω when each ω-isometry of K^n is the restriction of a $U(\omega)$-monomial map.

Extensibility Property (recall)

The symmetry group of ω.

$$U(\omega) = \{\lambda \in K^\times \mid \forall x \in K,\ \omega(\lambda x) = \omega(x)\} \leqslant K^\times$$

Extension Property

We say the extension property holds for the weight ω when each ω-isometry of K^n is the restriction of a $U(\omega)$-monomial map.

From Goldberg Theorem, one gets a criterion.

A sufficient condition for Extension Property

$$\mathrm{w}_\omega(x) = \sum_{i=1}^{n} \omega(x_i) = \sum_{r\in R} \omega(r) c_r(x) = \sum_{r\in\Omega} \omega(r) c_r^U(x).$$

A sufficient condition for Extension Property

$$\mathrm{w}_\omega(x) = \sum_{i=1}^{n} \omega(x_i) = \sum_{r\in R} \omega(r) c_r(x) = \sum_{r\in\Omega} \omega(r) c_r^U(x).$$

For all $s \in \Omega$,

$$\begin{aligned}\mathrm{w}_\omega(xs) &= \sum_{r\in R} \omega(rs) c_r(x)\\ &= \sum_{r\in\Omega} \omega(rs) c_r^U(x)\end{aligned}$$

A sufficient condition for Extension Property

$$\mathrm{w}_\omega(x) = \sum_{i=1}^{n} \omega(x_i) = \sum_{r\in R} \omega(r) c_r(x) = \sum_{r\in\Omega} \omega(r) c_r^U(x).$$

For all $s \in \Omega$,

$$\mathrm{w}_\omega(xs) = \sum_{r\in R} \omega(rs) c_r(x) = \sum_{r\in\Omega} \omega(rs) c_r^U(x)$$

idem,

$$\mathrm{w}_\omega(f(xs)) = \mathrm{w}_\omega(f(x)s) = \sum_{r\in\Omega} \omega(rs) c_r^U(f(x))$$

A sufficient condition for Extension Property

$$\mathrm{w}_\omega(x) = \sum_{i=1}^{n} \omega(x_i) = \sum_{r\in R} \omega(r) c_r(x) = \sum_{r\in\Omega} \omega(r) c_r^U(x).$$

For all $s \in \Omega$,

$$\mathrm{w}_\omega(xs) = \sum_{r\in R} \omega(rs) c_r(x) = \sum_{r\in\Omega} \omega(rs) c_r^U(x)$$

idem,

$$\mathrm{w}_\omega(f(xs)) = \mathrm{w}_\omega(f(x)s) = \sum_{r\in\Omega} \omega(rs) c_r^U(f(x))$$

Lemma

The invertibility of $\big(\omega(rs)\big)_{r,s\in\Omega}$ implies the U-preservation of ω whence Extension Property.

determinantal criterion

Let Ω a set of repretentatives for the action of $U := U(\omega)$.

$$\mathcal{W}_\omega := \begin{vmatrix} & \vdots & \\ \dots & \omega(rs) & \dots \\ & \vdots & \end{vmatrix}_{r,s\in\Omega} \qquad \Delta_\omega := \det(\mathcal{W}_\omega)$$

determinantal criterion

Let Ω a set of repretentatives for the action of $U := U(\omega)$.

$$\mathcal{W}_\omega := \begin{vmatrix} & \vdots & \\ \dots & \omega(rs) & \dots \\ & \vdots & \end{vmatrix}_{r,s\in\Omega} \qquad \Delta_\omega := \det(\mathcal{W}_\omega)$$

Proposition (Wood)

If $\Delta_\omega \neq 0$ then Extension Property holds for the weight ω.

determinantal criterion

Let Ω a set of repretentatives for the action of $U := U(\omega)$.

$$\mathcal{W}_\omega := \begin{vmatrix} & \vdots & \\ \dots & \omega(rs) & \dots \\ & \vdots & \end{vmatrix}_{r,s\in\Omega} \qquad \Delta_\omega := \det(\mathcal{W}_\omega)$$

Proposition (Wood)

If $\Delta_\omega \neq 0$ then Extension Property holds for the weight ω.

Remark

One has an analogue criterion non commutative case.

determinantal criterion

Let Ω a set of repretentatives for the action of $U := U(\omega)$.

$$\mathcal{W}_\omega := \begin{vmatrix} & \vdots & \\ \dots & \omega(rs) & \dots \\ & \vdots & \end{vmatrix}_{r,s\in\Omega} \qquad \Delta_\omega := \det(\mathcal{W}_\omega)$$

Proposition (Wood)

If $\Delta_\omega \neq 0$ then Extension Property holds for the weight ω.

Remark

One has an analogue criterion non commutative case.

Numerical evidence for the Lee weight!

Sommaire

Fourier coefficient

ℓ an odd prime

Fourier coefficient

ℓ an odd prime

- L the Lee metric of $\mathbb{F}_\ell$

Fourier coefficient

ℓ an odd prime

- L the Lee metric of $\mathbb{F}_\ell$
- $U(\mathrm{L}) = \{-1, +1\}$

Fourier coefficient

ℓ an odd prime

- L the Lee metric of $\mathbb{F}_\ell$
- $U(\mathrm{L}) = \{-1, +1\}$
- $G := \Omega = \mathbb{F}_\ell/\{-1, +1\}$ is cyclic of order $\frac{\ell-1}{2}$.

Fourier coefficient

ℓ an odd prime

- L the Lee metric of $\mathbb{F}_\ell$
- $U(\mathrm{L}) = \{-1, +1\}$
- $G := \Omega = \mathbb{F}_\ell/\{-1, +1\}$ is cyclic of order $\frac{\ell-1}{2}$.
- $\widehat{G}$ is the group of even (multiplicative) characters i.e. $\chi(-1) = 1$.

Fourier coefficient

ℓ an odd prime

- L the Lee metric of $\mathbb{F}_\ell$
- $U(\mathrm{L}) = \{-1, +1\}$
- $G := \Omega = \mathbb{F}_\ell/\{-1, +1\}$ is cyclic of order $\frac{\ell-1}{2}$.
- $\widehat{G}$ is the group of even (multiplicative) characters i.e. $\chi(-1) = 1$.

By the Dedekind determinant formula

$$\Delta_{\mathrm{L}} = \pm \prod_{\chi \in \widehat{G}} \widehat{\mathrm{L}}(\chi)$$

Fourier coefficient

ℓ an odd prime

- L the Lee metric of $\mathbb{F}_\ell$
- $U(\mathrm{L}) = \{-1, +1\}$
- $G := \Omega = \mathbb{F}_\ell/\{-1, +1\}$ is cyclic of order $\frac{\ell-1}{2}$.
- $\widehat{G}$ is the group of even (multiplicative) characters i.e. $\chi(-1) = 1$.

By the Dedekind determinant formula

$$\Delta_{\mathrm{L}} = \pm \prod_{\chi \in \widehat{G}} \widehat{\mathrm{L}}(\chi)$$

where $\widehat{\mathrm{L}}(\chi) = \sum_{s \in G} \mathrm{L}(s)\chi(s)$ is the Fourier coefficient of L at χ.

Sophie Germain case

Proposition

Certainly, Extension Property holds for the Lee weight in the case of sure prime module i.e. $\ell = 2p + 1$ *with* p *prime.*

Sophie Germain case

Proposition

Certainly, Extension Property holds for the Lee weight in the case of sure prime module i.e. $\ell = 2p + 1$ *with* p *prime.*

- β a generator of G

Sophie Germain case

Proposition

Certainly, Extension Property holds for the Lee weight in the case of sure prime module i.e. $\ell = 2p + 1$ with p prime.

- β a generator of G
- χ a non trivial character

Sophie Germain case

Proposition

Certainly, Extension Property holds for the Lee weight in the case of sure prime module i.e. $\ell = 2p + 1$ *with* p *prime.*

- β a generator of G
- χ a non trivial character
- $\zeta := \chi(\beta)$ is a primitive p-th root of unity.

Sophie Germain case

Proposition

Certainly, Extension Property holds for the Lee weight in the case of sure prime module i.e. $\ell = 2p + 1$ *with* p *prime.*

- β a generator of G
- χ a non trivial character
- $\zeta := \chi(\beta)$ is a primitive p-th root of unity.

The minimal polynomial of ζ is

$$\Phi_p(T) = T^{p-1} + \ldots + T^1 + T^0$$

Sophie Germain case

Proposition

Certainly, Extension Property holds for the Lee weight in the case of sure prime module i.e. $\ell = 2p + 1$ with p prime.

- β a generator of G
- χ a non trivial character
- $\zeta := \chi(\beta)$ is a primitive p-th root of unity.

The minimal polynomial of ζ is

$$\Phi_p(T) = T^{p-1} + \ldots + T^1 + T^0$$

thus

$$\widehat{\mathrm{L}}(\chi) = \sum_{k=0}^{p-1} \mathrm{L}(\beta^k)\zeta^k$$

does not vanish simply because L is not constant on G.

Two in one

We consider the Lee and Euclidean weights :

$$\mathrm{L}(t) = \begin{cases} t, & 0 \le t \le \ell/2; \\ \ell - t, & \ell/2 < t < \ell; \end{cases} \qquad \mathrm{E}(t) = \mathrm{L}(t)^2.$$

they share the same symmetry group

$$U := U(\mathrm{L}) = \{-1, +1\} = U(\mathrm{E}).$$

Theorem

If ℓ is an odd prime then $\Delta_{\mathrm{L}} \neq 0$ and $\Delta_{\mathrm{E}} \neq 0$.

Fourier coefficient of the Lee map

The quotient group

$$G := \mathbb{F}_{\ell}^{\times}/\{\pm 1\} = \{1, 2, \ldots, (\ell-1)/2\}$$

is cyclic of order $n := (\ell - 1)/2$.
we want to prove :

$$\forall \chi \in \widehat{G}, \quad 0 \neq \widehat{\mathrm{L}}(\chi) = \sum_{s \in G} \mathrm{L}(s)\chi(s).$$

- It is trivial when $\ell = 2p + 1$, p prime.
- Barra proved the case $\ell = 4p + 1$.

Fourier analysis

We identify $\widehat{G}$ with the group of even characters of $\mathbb{F}_\ell$:

$$\widehat{G} = \{\chi \in \widehat{\mathbb{F}_\ell^\times} \mid \chi(-1) = 1\}$$

The Fourier coefficients of L and E are given by

$$\widehat{\mathrm{L}}(\chi) = \sum_{x \in G} \mathrm{L}(x)\chi(x) = \sum_{k<\ell/2} \mathrm{L}(k)\chi(k) = \sum_{k<\ell/2} k\chi(k)$$

$$\widehat{\mathrm{E}}(\chi) = \sum_{x \in G} \mathrm{E}(x)\chi(x) = \sum_{k<\ell/2} \mathrm{E}(k)\chi(k) = \sum_{k<\ell/2} k^2\chi(k)$$

Links between the determinants

It is easy to verify the following quadratic relation holds

$$\mathrm{L}(2x)^2 - 4\mathrm{L}(x)^2 = \big(\mathrm{L}(2x) - 2\mathrm{L}(x)\big)\,\ell.$$

In other words

$$\mathrm{E}(2x) - 4\mathrm{E}(x) = \big(\mathrm{L}(2x) - 2\mathrm{L}(x)\big)\,\ell.$$

On spectra

$$(\bar{\chi}(2) - 4)\,\widehat{\mathrm{E}}(\chi) = (\bar{\chi}(2) - 2)\,\widehat{\mathrm{L}}(\chi)\,\ell.$$

Links between the determinants

It is easy to verify the following quadratic relation holds

$$\mathrm{L}(2x)^2 - 4\mathrm{L}(x)^2 = \big(\mathrm{L}(2x) - 2\mathrm{L}(x)\big)\,\ell.$$

In other words

$$\mathrm{E}(2x) - 4\mathrm{E}(x) = \big(\mathrm{L}(2x) - 2\mathrm{L}(x)\big)\,\ell.$$

On spectra

$$(\bar{\chi}(2) - 4)\,\widehat{\mathrm{E}}(\chi) = (\bar{\chi}(2) - 2)\,\widehat{\mathrm{L}}(\chi)\,\ell.$$

Scholie

Let r be the smallest positive integer such that $2^r \equiv \pm 1 \mod \ell$.

$$(2^r + 1)^{\frac{\ell-1}{2r}}\ \Delta_{\mathrm{E}} = \ell^{\frac{\ell-1}{2}}\ \Delta_{\mathrm{L}}.$$

basic fact for non trivial even characters

- $1 \neq \chi$ even and not trivial

$$\widehat{1}(\chi) = 2 \sum_{k<\ell/2} \chi(k) = 0.$$

basic fact for non trivial even characters

- $1 \neq \chi$ even and not trivial

$$\widehat{1}(\chi) = 2 \sum_{k<\ell/2} \chi(k) = 0.$$

The first generalized Bernoulli's number vanishes too

$$B_1(\chi) = \frac{1}{\ell} \sum_{k=1}^{\ell} k\chi(k) = 0$$

basic fact for non trivial even characters

- $1 \neq \chi$ even and not trivial

$$\widehat{1}(\chi) = 2 \sum_{k<\ell/2} \chi(k) = 0.$$

The first generalized Bernoulli's number vanishes too

$$B_1(\chi) = \frac{1}{\ell} \sum_{k=1}^{\ell} k\chi(k) = 0$$

We want to prove that

$$0 \neq \frac{1}{\ell} \sum_{k<\ell/2} k\chi(k) = \widehat{\mathrm{L}}(\chi)$$

Consequence of $\widehat{\mathrm{L}}(\chi) = 0$ on the 2nd Bernoulli's number

Let us observe the consequence of

$$\widehat{\mathrm{L}}(\chi) = 0 = \widehat{\mathrm{E}}(\chi), \quad 1 \neq \chi, \quad \chi(-1) = 1,$$

on the second generalized Bernoulli's number

$$B_2(\chi) = \frac{1}{2\ell} \sum_{k=1}^{\ell} (k^2 - \ell k)\chi(k).$$

$$\begin{aligned} 2\ell B_2(\chi) &= 2\widehat{\mathrm{E}}(\chi) - 2\widehat{\mathrm{L}}(\chi)\ell + \widehat{1}(\chi)\ell^2 \\ &= \textit{zero}. \end{aligned}$$

Consequence of $\widehat{\mathrm{L}}(\chi) = 0$ on the 2nd Bernoulli's number

Let us observe the consequence of

$$\widehat{\mathrm{L}}(\chi) = 0 = \widehat{\mathrm{E}}(\chi), \quad 1 \neq \chi, \quad \chi(-1) = 1,$$

on the second generalized Bernoulli's number

$$B_2(\chi) = \frac{1}{2\ell} \sum_{k=1}^{\ell} (k^2 - \ell k)\chi(k).$$

$$2\ell B_2(\chi) = 2\widehat{\mathrm{E}}(\chi) - 2\widehat{\mathrm{L}}(\chi)\ell + \widehat{1}(\chi)\ell^2$$
$$= \textit{zero}.$$

Contradiction with classical fact from number theory

In number theory, there is a long story concerning the analytic continuation of the Dirichlet serie

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$$

Contradiction with classical fact from number theory

In number theory, there is a long story concerning the analytic continuation of the Dirichlet serie

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$$

On the one hand

$$-B_2(\chi)/2 = L(-1,\chi)$$

Contradiction with classical fact from number theory

In number theory, there is a long story concerning the analytic continuation of the Dirichlet serie

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$$

On the one hand

$$-B_2(\chi)/2 = L(-1,\chi)$$

On the other hand

$$L(-1,\chi)=0 \text{ if and only if } \chi \text{ is odd.}$$

Contradiction with classical fact from number theory

In number theory, there is a long story concerning the analytic continuation of the Dirichlet serie

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$$

On the one hand

$$-B_2(\chi)/2=L(-1,\chi)$$

On the other hand

$$L(-1,\chi)=0 \text{ if and only if } \chi \text{ is odd.}$$

whence the determinants Δ_{L} and Δ_{E} do not vanish.

Contradiction with classical fact from number theory

In number theory, there is a long story concerning the analytic continuation of the Dirichlet serie

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$$

On the one hand

$$-B_2(\chi)/2=L(-1,\chi)$$

On the other hand

$$L(-1,\chi)=0 \text{ if and only if } \chi \text{ is odd.}$$

whence the determinants Δ_{L} and Δ_{E} do not vanish.

primary case

Corollary (extension property)

The Lee and Euclidean isometries are the restriction of $\{-1, +1\}$*-monomial transformations.*

primary case

Corollary (extension property)

The Lee and Euclidean isometries are the restriction of $\{-1, +1\}$-monomial transformations.

The same approach works in the case of a primary module

primary case

Corollary (extension property)

The Lee and Euclidean isometries are the restriction of $\{-1,+1\}$*-monomial transformations.*

The same approach works in the case of a primary module

but not for a composite module!

Sommaire

Additive Fourier coefficient

The additive Fourier coefficient of ω :

$$\omega^\star(a) = \sum_{x\in\mathbb{F}_\ell} \omega(x)\mu(ax)$$

where μ is the standard additive character of $\mathbb{F}_\ell$.

Additive Fourier coefficient

The additive Fourier coefficient of ω :

$$\omega^\star(a) = \sum_{x \in \mathbb{F}_\ell} \omega(x)\mu(ax)$$

where μ is the standard additive character of $\mathbb{F}_\ell$.

Note that $U(\omega^\star) = U(\omega)$ and

$$\sum_{a \in \mathbb{F}_\ell} \omega^\star(a) = \ell \times \omega(0) = 0$$

change of determinant

Since $\omega(0) = 0$,

$$\widehat{\omega^\star}(\chi) = \tau(\chi)\widehat{\omega}(\bar{\chi})$$

where $\tau(\chi)$ is a Gauss sum

$$\mathcal{W}_\omega^\star = \begin{vmatrix} & \vdots & \\ \dots & \omega^\star(rs) & \dots \\ & \vdots & \end{vmatrix}_{r,s \in \mathbb{F}_\ell^\times/\pm 1}$$

change of determinant

Since $\omega(0) = 0$,

$$\widehat{\omega^\star}(\chi) = \tau(\chi)\widehat{\omega}(\bar{\chi})$$

where $\tau(\chi)$ is a Gauss sum

$$\mathcal{W}^\star_\omega = \begin{vmatrix} & \vdots & \\ \dots & \omega^\star(rs) & \dots \\ & \vdots & \end{vmatrix}_{r,s\in\mathbb{F}_\ell{}^\times/\pm 1}$$

$$\Delta_\omega = 0 \Leftrightarrow \det(\mathcal{W}^\star_\omega) = 0$$

Levy-Desplanques dominant criterion

A strictly diagonally dominant $n \times n$-matrix (a_{ij}) i.e.

$$\forall i, \quad |a_{ii}| > \sum_{i \neq j} |a_{ij}|$$

is not singular.

Corollary

If

$$\forall r \neq 0, \quad \omega^\star(r) < 0 \qquad \textit{and} \qquad \omega^\star(0) < -2\,|U(\omega)| \times \omega^\star(1)$$

then $\Delta_\omega \neq 0$.

sketch

We consider the matrices

$$\mathcal{W}_\omega^\star = \begin{vmatrix} & \vdots & \\ \dots & \omega^\star(rs) & \dots \\ & \vdots & \end{vmatrix}_{r,s\in\mathbb{F}_\ell{}^\times/\pm 1} \qquad \begin{vmatrix} & \vdots & \\ \dots & \omega^\star(r/s) & \dots \\ & \vdots & \end{vmatrix}_{r,s\in\mathbb{F}_\ell{}^\times/\pm 1}$$

$\omega^\star(1)$ is on the diagonal

sketch

We consider the matrices

$$\mathcal{W}_\omega^\star = \begin{vmatrix} & \vdots & \\ \dots & \omega^\star(rs) & \dots \\ & \vdots & \end{vmatrix}_{r,s\in\mathbb{F}_\ell{}^\times/\pm 1} \qquad \begin{vmatrix} & \vdots & \\ \dots & \omega^\star(r/s) & \dots \\ & \vdots & \end{vmatrix}_{r,s\in\mathbb{F}_\ell{}^\times/\pm 1}$$

$\omega^\star(1)$ is on the diagonal

$$|\omega^\star(1)| - \sum_{1\neq r\in\Omega} |\omega^\star(r)| = -\omega^\star(1) + \sum_{1\neq r\in\Omega} \omega^\star(r)$$

sketch

We consider the matrices

$$\mathcal{W}_\omega^\star = \begin{vmatrix} & \vdots & \\ \dots & \omega^\star(rs) & \dots \\ & \vdots & \end{vmatrix}_{r,s\in\mathbb{F}_\ell{}^\times/\pm 1} \qquad \begin{vmatrix} & \vdots & \\ \dots & \omega^\star(r/s) & \dots \\ & \vdots & \end{vmatrix}_{r,s\in\mathbb{F}_\ell{}^\times/\pm 1}$$

$\omega^\star(1)$ is on the diagonal

$$|\omega^\star(1)| - \sum_{1\neq r\in\Omega} |\omega^\star(r)| = -\omega^\star(1) + \sum_{1\neq r\in\Omega} \omega^\star(r)$$

The sum of the Fourier coefficients $\omega^\star(0) + \sharp U(\omega) \times \sum_{r\in\Omega} \omega^\star(r)$ vanishes.

sketch

We consider the matrices

$$\mathcal{W}^\star_\omega = \begin{vmatrix} & \vdots & \\ \dots & \omega^\star(rs) & \dots \\ & \vdots & \end{vmatrix}_{r,s\in\mathbb{F}_\ell{}^\times/\pm 1} \qquad \begin{vmatrix} & \vdots & \\ \dots & \omega^\star(r/s) & \dots \\ & \vdots & \end{vmatrix}_{r,s\in\mathbb{F}_\ell{}^\times/\pm 1}$$

$\omega^\star(1)$ is on the diagonal

$$|\omega^\star(1)| - \sum_{1\neq r\in\Omega} |\omega^\star(r)| = -\omega^\star(1) + \sum_{1\neq r\in\Omega} \omega^\star(r)$$

The sum of the Fourier coefficients $\omega^\star(0) + \sharp U(\omega) \times \sum_{r\in\Omega} \omega^\star(r)$ vanishes.

$$|\omega^\star(1)| - \sum_{1\neq r\in\Omega} |\omega^\star(r)| = -2\omega^\star(1) + \frac{-\omega^\star(0)}{\sharp U(\omega)}$$

Additive Fourier coefficient of the Lee map

- $0 \leq r < \ell$
- $n := \frac{\ell-1}{2}$
- $t := \frac{2\pi r}{\ell}$

Additive Fourier coefficient of the Lee map

- $0 \leq r < \ell$
- $n := \frac{\ell-1}{2}$
- $t := \frac{2\pi r}{\ell}$

$$\mathrm{L}^{\star}(r) = \sum_{k=0}^{\ell-1} \mathrm{L}(k) e^{it} = 2 \sum_{k=1}^{n} \cos kt = 2n \left(D_n(t) - F_n(t) \right)$$

where D_n is the Dirichlet kernel

Additive Fourier coefficient of the Lee map

- $0 \leq r < \ell$
- $n := \frac{\ell-1}{2}$
- $t := \frac{2\pi r}{\ell}$

$$\mathrm{L}^\star(r) = \sum_{k=0}^{\ell-1} \mathrm{L}(k) e^{it} = 2\sum_{k=1}^{n} \cos kt = 2n\,(D_n(t) - F_n(t))$$

where D_n is the Dirichlet kernel

$$D_n(t) := \frac{1}{2} + \sum_{k=-n}^{n} e^{ikt} = \frac{1}{2} + \sum_{k=1}^{n} \cos kt = \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t} = 0$$

and F_n the Fejér kernel

Additive Fourier coefficient of the Lee map

- $0 \le r < \ell$
- $n := \frac{\ell-1}{2}$
- $t := \frac{2\pi r}{\ell}$

$$\mathrm{L}^\star(r) = \sum_{k=0}^{\ell-1} \mathrm{L}(k) e^{it} = 2\sum_{k=1}^{n} \cos kt = 2n\left(D_n(t) - F_n(t)\right)$$

where D_n is the Dirichlet kernel

$$D_n(t) := \frac{1}{2} + \sum_{k=-n}^{n} e^{ikt} = \frac{1}{2} + \sum_{k=1}^{n} \cos kt = \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t} = 0$$

and F_n the Fejér kernel

$$F_n(t) := \frac{1}{n}\sum_{k=0}^{n-1} D_k(t) = \frac{1}{2} + \frac{1}{n}\sum_{k=1}^{n}(1-\frac{k}{n})\cos kt = \frac{1}{2n}\left(\frac{\sin\frac{n}{2}t}{\sin\frac{1}{2}t}\right)^2$$

Lee weight satisfies the two conditions

- $0 \leq r < \ell$
- $n := \frac{\ell-1}{2}$

First condition :

$$\mathrm{L}^\star(r) = -2nF_n(\frac{2\pi r}{\ell}) < 0$$

Second condition :

$$-4\mathrm{L}^\star(1) = 4\left(\frac{\sin\frac{\frac{\ell-1}{2}}{2}\frac{2\pi}{\ell}}{\sin\frac{1}{2}\frac{2\pi}{\ell}}\right)^2$$

and

$$\mathrm{L}^\star(0) = 2\sum_{k=1}^{\frac{\ell-1}{2}} k = \frac{1}{4}(\ell^2 - 1)$$

second condition

We have to prove

$$-4\mathrm{L}^\star(1) \quad > \quad \mathrm{L}^\star(0)$$

second condition

We have to prove

$$-4\mathrm{L}^{\star}(1) \quad > \quad \mathrm{L}^{\star}(0)$$

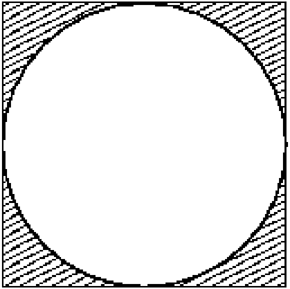
and now it is very easy !

second condition

We have to prove

$$-4\mathrm{L}^{\star}(1) \quad > \quad \mathrm{L}^{\star}(0)$$

and now it is very easy !

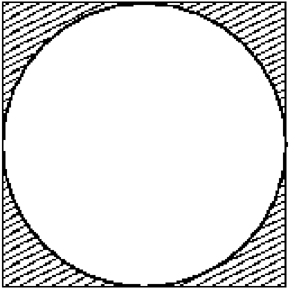


second condition

We have to prove

$$-4\mathrm{L}^\star(1) \quad > \quad \mathrm{L}^\star(0)$$

and now it is very easy !



Indeed,

$$\frac{4}{\pi^2}\ell^2 \sim -4\mathrm{L}^\star(1) \quad \text{and} \quad \mathrm{L}^\star(0) \sim \frac{1}{4}\ell^2$$

Dyshko criterion for modular ring

- consider the ring $\mathbb{Z}/(q)$
- ω a weight function
- b a divisor of q.
- write $q = ab$

Dyshko criterion for modular ring

- consider the ring $\mathbb{Z}/(q)$
- ω a weight function
- b a divisor of q.
- write $q = ab$

Consider the additive Fourier coefficients of the map $x \mapsto \omega(bx)$

$$F_a(t) = \sum_{x \in \mathbb{Z}/(a)} \omega(bx)\zeta_a^{tx}$$

Dyshko criterion for modular ring

- consider the ring $\mathbb{Z}/(q)$
- ω a weight function
- b a divisor of q.
- write $q = ab$

Consider the additive Fourier coefficients of the map $x \mapsto \omega(bx)$

$$F_a(t) = \sum_{x \in \mathbb{Z}/(a)} \omega(bx)\zeta_a^{tx}$$

$$W_a(\omega) = \begin{vmatrix} & \vdots & \\ \dots & \omega^\star(rs) & \dots \\ & \vdots & \end{vmatrix}_{r,s \in \mathbb{Z}/(a)^*/G_a(\omega)}$$

where $G_a(\omega) = \{h \in \mathbb{Z}/(a)^* \mid \forall t \in \mathbb{Z}/(a) \quad wt(bht) = \omega(bt)\}$.

Dyshko criterion for modular ring

Theorem (Dyshko)

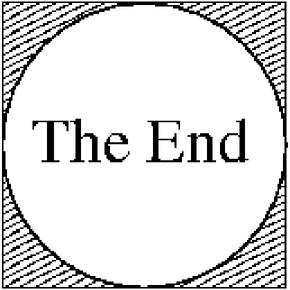
Let $\omega\colon \mathbb{Z}/(q) \to \mathbb{C}$ be a weight function. If for all $1 \neq a \mid q$ the matrix $W_a(\omega)$ is non singular and

$$\forall h \in G_a(\omega) \quad \exists g \in G_q(\omega) \quad g \equiv h \mod a$$

then Extension Property holds for the weight ω.

Corollary

For every integer $q \geq 2$ the Extension Property of the Lee weight holds over the ring $\mathbb{Z}/(q)$.


The End